University of California, Berkeley
Physics 110B, Spring 2004 (*Strovink*)

HINTS FOR ASSIGNED EXERCISES 26-41

26.
The field strength and dual field strength tensors are

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$
$$G^{\mu\nu} \equiv \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \ ,$$

where $\epsilon^{\mu\nu\rho\sigma} = 1\ (-1)$ when $\mu\nu\rho\sigma$ are even (odd) permutations of 0123, and 0 otherwise. You may use their explicit elements as given in SCSR Eqs. (62) and (65).
(a.)
By explicit evaluation, show that $F^{\mu\nu}F_{\mu\nu}$ is proportional to $E^2 - c^2B^2$, and find the constant of proportionality. (Because $F^{\mu\nu}F_{\mu\nu}$ is obviously a Lorentz scalar, the Lorentz invariance of $E^2 - c^2B^2$ is therefore said to be *manifest*.)
Hint:
Start with SCSR Eq. (62) for the $F^{\mu\nu}$. Use SCSR Eqs. (58) (or the information in the top right quarter of SCSR page 17) to get the $F_{\mu\nu}$. Now,

$$F^{\mu\nu}F_{\mu\nu} = F^{00}F_{00} + F^{01}F_{01} + \ldots + F^{33}F_{33}$$

(16 terms). Add up the terms.
(b.)
By explicit evaluation, show that $F^{\mu\nu}G_{\mu\nu}$ is proportional to $\vec{E}\cdot\vec{B}$, and find the constant of proportionality. (Likewise the Lorentz invariance of $\vec{E}\cdot\vec{B}$ is manifest.)
Hint:
Start with SCSR Eq. (65) for the $G^{\mu\nu}$.
(c.)
What two criteria must (uniform nonzero) $\vec{E}$ and $\vec{B}$ satisfy in the lab frame so that, in a different inertial frame, $\vec{B}$ is allowed to vanish?
Hint:
If $\vec{B}$ is to vanish in one Lorentz frame, can $\vec{E}\cdot\vec{B}$ (a Lorentz invariant) be nonzero in any Lorentz frame? If $|\vec{E}|$ is to exceed $c|\vec{B}|$ in one Lorentz frame, can $E^2 - c^2B^2$ (a Lorentz invariant) be negative or zero in any Lorentz frame?

27.
Griffiths Problem 12.36.
Hint:
Using the notation employed in class, and dividing through by mc, Griffiths asks you to prove

$$\frac{d}{dt}\gamma\vec{\beta} = \gamma\big(\dot{\vec{\beta}} + \gamma^2\vec{\beta}(\vec{\beta}\cdot\dot{\vec{\beta}})\big) \ .$$

Writing γ in terms of β and routinely differentiating the LHS, cancelling the term $\gamma\dot{\vec{\beta}}$ on both sides, and ignoring the common factor $\gamma^3\vec{\beta}$, this reduces to proving

$$\beta\frac{d\beta}{dt} = \vec{\beta}\cdot\frac{d\vec{\beta}}{dt} \ .$$

To do this, consider $\frac{d}{dt}(\vec{\beta}\cdot\vec{\beta})$.

28.
You are an indefagitable runner of rest mass m, whose feet generate a constant force F_0 in the x direction (as observed in the *laboratory*). This force causes your (relativistic) momentum to increase linearly with laboratory time t.
(a.)
At $t = 0$, when you are at rest at the origin, your feet begin to exert this force. Thereafter, show that $\sinh\eta$, where η is your rapidity, is proportional to t, and find the constant of proportionality.
Hint:
Along $\hat{x}$, the time derivative of

$$p = \gamma\beta mc = \cosh\eta\tanh\eta\, mc$$

is equal to F_0.
(b.)
At $t = t_1$, a laser pulse is shot from the origin in the x direction. How much of a head start t_1 do you require in order for the laser pulse never to

catch up with you?

Hint:

From part (a.) you know that $\sinh\eta = \omega t$, where ω is the constant of proportionality that you obtained there. Using $\cosh^2\eta - \sinh^2\eta = 1$, you also know the time dependence of $\cosh\eta$. Taking the ratio, you know the time dependence of $\beta = \tanh\eta$. Integrate $c\beta(t)$ (a perfect differential) to obtain the runner's equation of motion

$$\tfrac{\omega}{c}x(t) = \sqrt{1+\omega^2 t^2} - 1 \ .$$

Compare this to the laser pulse's equation of motion

$$\tfrac{\omega}{c}x_p(t) = \omega(t - t_1) \ .$$

For what value of t_1 will x stay larger than x_p (infinitesimally so when ωt is $\gg 1$ but finite)?

29.

At $t = 0$, a particle of rest mass m and charge e is at rest at the origin. It accelerates under the influence of a uniform static electric field $\vec{E} = \hat{z}E_0$.

(a.)

For $t > 0$, show that the relativistic solution for $z(t)$ is given by

$$\begin{aligned} z(t) &= \int_0^t c\beta_z(t)\,dt \\ \beta_z(t) &= \tanh\eta(t) \\ \eta(t) &= \sinh^{-1}\frac{eE_0}{mc}t \ . \end{aligned}$$

Hint:

Use the hint for part (a.) of the previous problem.

(b.)

Suppose instead that, at $t = 0$, the particle has an initial momentum $\vec{p}(0) = \hat{x}p_\perp$. Show that the solution for $z(t)$ is the same as in (a.), except that m is replaced by m_{eff}, where

$$m_{\text{eff}} = \sqrt{m^2 + p_\perp^2/c^2} \ .$$

This says that, under the influence of a uniform electrostatic field, the longitudinal motion of a particle with nonzero transverse momentum is the same as that of a heavier particle with zero transverse momentum.

Hint:

Why is $p_\perp$ constant? When $p_\perp = 0$, the equation of motion along $\hat{z}$ is determined by

$$\frac{d}{dt}\gamma\beta mc = eE_0 \ ,$$

where

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} = \frac{E}{mc^2}$$

and E is the particle's total energy. When $\vec{p}_\perp \neq 0$, if you can show that

$$\frac{d}{dt}\gamma_{\text{eff}}\beta_z m_{\text{eff}}c = eE_0 \ , \qquad \text{(A)}$$

where

$$\gamma_{\text{eff}} \equiv \frac{E}{m_{\text{eff}}c^2} = \frac{1}{\sqrt{1-\beta_z^2}} \ , \qquad \text{(B)}$$

the assertion to be proved will be justified. To prove Eq. (A), consider the force equation $dp_z/dt = eE_0$ and use the fact that $\gamma m = \gamma_{\text{eff}}m_{\text{eff}}$. To prove Eq. (B), write $\beta_z = cp_z/E$.

(c.)

Under the conditions of part (b.), for $t > 0$ does $x(t)$ increase linearly with t? Explain.

Hint:

You have argued that $p_\perp = \gamma\beta_x mc$ is constant. So β_x is constant if γ is constant.

30.

At $t = 0$, a particle of rest mass m and charge e is at the origin, with initial momentum $\vec{p}(0) = \hat{x}p_\perp$. It accelerates under the influence of a uniform static magnetic field $\vec{B} = \hat{z}B_0$.

(a.)

For $t > 0$, show that the relativistic solution for $x(t)$ and $y(t)$ is given by

$$\begin{aligned} x(t) &= \int_0^t c\beta_x(t)\,dt \\ y(t) &= \int_0^t c\beta_y(t)\,dt \\ \gamma_\perp mc\beta_x(t) &= p_\perp\cos\omega_c t \\ \gamma_\perp mc\beta_y(t) &= p_\perp\sin\omega_c t \\ \omega_c &= -\frac{eB_0}{\gamma_\perp m} \\ \gamma_\perp m &= \sqrt{m^2 + p_\perp^2/c^2} \ . \end{aligned}$$

Hint:
First, by considering the power $\vec{F} \cdot c\vec{\beta}$ and the nature of the Lorentz force in a pure magnetic field, show that $|\vec{p}_\perp|$ and therefore $\gamma_\perp$ is constant. Show that the Lorentz force equation can be written

$$\frac{d\vec{\beta}}{dt} = (\hat{z}\omega_c) \times \vec{\beta}\,.$$

This is an equation for (CCW) precession of $\vec{\beta}$ about $\hat{z}$ with angular frequency ω_c. (Considering the negative sign of ω_c, the precession is actually CW).

(b.)
Suppose instead that, at $t = 0$, the particle has an initial momentum $\vec{p}(0) = \hat{x}p_\perp + \hat{z}p_\parallel$. Show that the solution for $x(t)$ and $y(t)$ is the same as in (a.), except that $\gamma_\perp$ is replaced by γ, where

$$\gamma m = \sqrt{m^2 + p_\perp^2/c^2 + p_\parallel^2/c^2}\,.$$

Hint:
Show that the solution is the same as in part (a), except that

$$\omega_c = -\frac{eB_0}{\gamma m}\,.$$

(c.)
Show that the result of (b.) alternatively can be expressed as the result of (a.) with m replaced by

$$m_{\text{eff}} = \sqrt{m^2 + p_\parallel^2/c^2}\,.$$

This says that, under the influence of a uniform magnetostatic field, the transverse motion of a particle with nonzero longitudinal momentum is the same as that of a heavier particle with zero longitudinal momentum.

Hint:
Show that

$$\gamma m = \sqrt{m_{\text{eff}}^2 + p_\parallel^2/c^2}\,.$$

(d.)
Under the conditions of part (b.), for $t > 0$ does $z(t)$ increase linearly with t? Explain.

Hint:
Use the fact that $p_\parallel = \gamma\beta_z mc$. Are $p_\parallel$ and γ both constant?

31.
At $t = 0$ at the origin of a spherical polar coordinate system in the lab, a point particle of charge q has velocity βc directed along the $\hat{z}$ (north polar) axis. It has been moving with that constant velocity for a long time.

(a.)
Starting with the Coulomb field in the particle's rest frame, and using the rules for relativistic transformation of EM fields, show that the electric field observed in the lab at $t = 0$ and $\vec{r} = (r, \theta, \phi)$ is

$$4\pi\epsilon_0\vec{E} = \hat{r}\frac{q}{r^2}\frac{\gamma}{(\gamma^2\cos^2\theta + \sin^2\theta)^{3/2}}\,,$$

where as usual $\gamma = 1/\sqrt{1-\beta^2}$.

Hint:
Work in Cartesian coordinates. Put the observer in frame $\mathcal{S}$ at $(x, 0, z)$, with $\hat{z}$ the direction of the charge's motion. Put the charge at the origin $(x', y', z') = (0, 0, 0)$ of frame $\mathcal{S}'$. Write E'_x and E'_z in terms of x', y', and z'. Then transform E'_x, E'_z, and z' to frame $\mathcal{S}$. Finally, convert to spherical polar coordinates in the lab frame.

(b.)
Show that this result is equivalent to Griffiths Eq. (10.68).

Hint:
In Griffiths' notation, $\mathbf{R}$ is the same as our $\vec{r}$. Remove a factor γ^3 from the denominator of the answer to part (a.), and substitute $\cos^2\theta = 1 - \sin^2\theta$.

32.
Griffiths Problem 10.9(b).

Hint:
Substitute $I(t_{\text{ret}}) = q_0\delta(t_{\text{ret}}) = q_0\delta(t - \mathcal{R}/c)$ in Griffiths' first equation of his solution to Example 10.2. Since the integrand is even, integrate from 0 to ∞ rather than from $-\infty$ to ∞. Using $z = \sqrt{\mathcal{R}^2 - s^2}$, where s is the cylindrical radial coordinate, convert the integral over dz to an integral over $d\mathcal{R}$. Convert the delta function $\delta(t - \mathcal{R}/c)$ to $\delta(\mathcal{R} - ct)$ using the rule $\delta(f(x)) = \delta(x)/|df/dx|$ (true if $f = 0$ when $x = 0$). The delta function trivializes the integration. Differentiate your answer for $\vec{A}$ to get $\vec{E}$ and $\vec{B}$.

33.
The general expression for the electromagnetic fields arising from a point particle of charge q moving with velocity $\vec{\beta}c$ and acceleration $\dot{\vec{\beta}}c$ is

$$\begin{aligned} \vec{E} &= \vec{E}_{\rm v} + \vec{E}_{\rm a} \ \text{ with} \\ \vec{E}_{\rm v} &= \frac{q}{4\pi\epsilon_0}\Big\{\frac{1}{\mathcal{R}^2}\frac{(\hat{\mathcal{R}}-\vec{\beta})(1-\beta^2)}{(1-\hat{\mathcal{R}}\cdot\vec{\beta})^3}\Big\}_{\rm ret} \\ c\vec{B} &= \big\{\hat{\mathcal{R}}\times\vec{E}\big\}_{\rm ret}\ , \end{aligned}$$

where $\vec{E}_{\rm v}$ is the *velocity field*, and the *acceleration field* $\vec{E}_{\rm a}$ is given in a later problem. Here $\vec{r}$ is a vector from the origin to the observer, $\vec{w}(t)$ is a vector from the origin to the particle, $\vec{\mathcal{R}} \equiv \vec{r}-\vec{w}$, and the subscript "ret" means that quantities are to be evaluated at time $t_{\rm ret} = t - \mathcal{R}/c$.

Assume that $\vec{\beta}$ lies in the z direction and is a constant, so that the acceleration field vanishes. As usual $\theta = \cos^{-1}\hat{z}\cdot\hat{r}$. Choose the origin of coordinates to be the position of the particle at $t = 0$. At that time, show that...
(a.)

$$-ct_{\rm ret} = \gamma\big(\gamma\beta z + \sqrt{(\gamma\beta z)^2 + r^2}\big)\ ;$$

Hint:
At $t = 0$,

$$ct_{\rm ret} = -\mathcal{R} = |\vec{r} - \hat{z}\beta ct_{\rm ret}|\ .$$

Solve this quadratic equation for $ct_{\rm ret}$ and choose the solution that yields $-ct_{\rm ret} = r$ when $\beta = 0$.
(b.)

$$\mathcal{R}(1-\hat{\mathcal{R}}\cdot\vec{\beta}) = r\sqrt{1-\beta^2\sin^2\theta}\ ;$$

Hint:
Write the LHS as $\mathcal{R} - \vec{\mathcal{R}}\cdot\vec{\beta}$. At $t = 0$, $\mathcal{R} = -ct_{\rm ret}$ (which is positive). Combining terms, you should obtain

$$\mathcal{R}(1-\hat{\mathcal{R}}\cdot\vec{\beta}) = -ct_{\rm ret}(1-\beta^2) - \beta z\ .$$

Substituting the result of part (a.) for $ct_{\rm ret}$ results in a lovely cancellation.
(c.)

$$\vec{r} = \mathcal{R}(\hat{\mathcal{R}} - \vec{\beta})\ .$$

Hint:
Write the RHS as $\vec{\mathcal{R}} - \mathcal{R}\vec{\beta}$. You have already encountered all the ingredients you need: at $t = 0$, $\vec{\mathcal{R}} = \vec{r} - \vec{\beta}ct_{\rm ret}$ and $ct_{\rm ret} = -\mathcal{R}$.

34.
Under the conditions of the previous problem, and using the tools developed there, show that $\vec{E}_{\rm v}$ is equivalent to Griffiths Eq. (10.68).
Hint:
Tools (b.) and (c.) from the previous problem are sufficient.

35.
Liénard's equation for the Poynting vector

$$\vec{S}_{\rm a} = \frac{1}{\mu_0}\vec{E}_{\rm a}\times\vec{B}_{\rm a}$$

arising from acceleration of a point particle of charge q is

$$\vec{S}_{\rm a} = (\frac{q}{4\pi\epsilon_0})^2\frac{\epsilon_0}{c}\Big\{\frac{\hat{\mathcal{R}}}{\mathcal{R}^2}\Big[\frac{\hat{\mathcal{R}}\times[(\hat{\mathcal{R}}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\hat{\mathcal{R}}\cdot\vec{\beta})^3}\Big]^2\Big\}_{\rm ret}.$$

(a.)
Show that Liénard's equation follows directly from the electric and magnetic fields arising from acceleration of a point particle, using the *acceleration fields*

$$\begin{aligned} \vec{E}_{\rm a} &= \frac{q}{4\pi\epsilon_0}\frac{1}{c}\Big\{\frac{1}{\mathcal{R}}\frac{\hat{\mathcal{R}}\times[(\hat{\mathcal{R}}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\hat{\mathcal{R}}\cdot\vec{\beta})^3}\Big\}_{\rm ret} \\ c\vec{B}_{\rm a} &= \big\{\hat{\mathcal{R}}\times\vec{E}_{\rm a}\big\}_{\rm ret}\ . \end{aligned}$$

Hint:
After applying the $bac - cab$ rule, only one term survives.
(b.)
Suppose that the particle is in uniform motion around a circle of radius b in the plane $z = 0$ centered at the origin. The motion is ultrarelativistic, *i.e.* $(1-\beta^2)^{-1/2} \gg 1$. To lowest order, calculate the radiated power per unit area observed at $(0, 0, z)$, where $z \gg b$.
Hint:
To an excellent approximation, $\hat{\mathcal{R}}$ is perpendicular to both $\vec{\beta}$ and $\dot{\vec{\beta}}$. This makes $\vec{S}_a$ trivial to evaluate.

(c.)
Is $\hat{z}$ a direction in which the power radiated per unit solid angle is near the maximum for this motion? Explain.
Hint:
Does the "train factor" $(1-\hat{\mathcal{R}}\cdot\vec{\beta})^{-6}$ provide any enhancement at this observation point?

36.
As an intermediate step in the derivation of the velocity and acceleration fields $\vec{E}_{\rm v}$ and $\vec{E}_{\rm a}$, in class we derived the expression

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\Big\{\frac{1}{1-\hat{\mathcal{R}}\cdot\vec{\beta}}\Big[\frac{\hat{\mathcal{R}}}{\mathcal{R}^2} + \frac{d}{c\,dt}\frac{\hat{\mathcal{R}}-\vec{\beta}}{\mathcal{R}(1-\hat{\mathcal{R}}\cdot\vec{\beta})}\Big]\Big\}_{\rm ret}$$

where the subscript "ret" means that the differentiation should be done first, and afterward all time-dependent quantities should be evaluated at time $t_{\rm ret} = t - \mathcal{R}/c$.

Define $\dot{\vec{\beta}} \equiv d\vec{\beta}/dt$. Use two relations worked out in class:

$$\frac{d\mathcal{R}}{c\,dt} = -\hat{\mathcal{R}}\cdot\vec{\beta}$$
$$\frac{d\hat{\mathcal{R}}}{c\,dt} = \frac{\hat{\mathcal{R}}\times(\hat{\mathcal{R}}\times\vec{\beta})}{\mathcal{R}}\ .$$

With these tools, finish the derivation to obtain $\vec{E}_{\rm v}$ (as given in an earlier problem) and $\vec{E}_{\rm a}$ (as given in the previous problem).
Hint:
For the terms involving $\dot{\vec{\beta}}$, apply the $bac-cab$ rule in reverse. For the terms not involving $\dot{\vec{\beta}}$, apply the $bac-cab$ rule directly to $\hat{\mathcal{R}}\times(\hat{\mathcal{R}}\times\vec{\beta})$.

37.
Griffiths Problem 3.40.
Hint:
Start with the definition

$$q_{lm} \equiv \int d\tau'\rho(\vec{r}\,')r'^l\,Y^*_{lm}(\theta',\phi')$$

of the multipole moments. A line charge is azimuthally symmetric, causing the $m\neq 0$ moments to vanish since

$$\int d\phi\, e^{im\phi} = 0\ .$$

For a line charge along the z axis, $\int\rho\,d\tau'$ reduces to $\int\lambda\,dz$.

38.
The electrostatic potential created by a static point charge can take a nontrivial form when the coordinate system is chosen to have an origin which, for some other reason, must be centered at point that does not coincide with the charge's position.

This problem concerns the potential $V(\vec{r})$ created by a localized charge distribution $\rho(\vec{r}\,')$. With the observation point located outside the charge distribution ($r > r'_{\rm max}$), use the standard expansion in spherical harmonics

$$\epsilon_0 V(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{Y_{lm}(\theta,\phi)}{(2l+1)r^{l+1}}\,q_{lm}\ ,$$

where the multipole moments q_{lm} are defined in the hint for the previous problem. In spherical polar coordinates, consider a point charge e located at (r',θ',ϕ') with respect to a certain origin. Determine the electrostatic potential that it creates at an observation point (r,θ,ϕ), with $r > r'_{\rm max}$.
(a.)
Write down the exact value of V$(r,\theta,\phi,r',\theta',\phi')$ as an infinite sum over l and m.
Hint:
The charge distribution for a point charge is a delta function that trivializes the q_{lm} integral. Compare your result to the expression in your lecture notes for the expansion in spherical harmonics of $1/|\vec{r}-\vec{r}\,'|$.
(b.)
Explicitly evaluating the spherical harmonics as functions of θ and ϕ (or θ' and ϕ'), write down all the monopole, dipole, and quadrupole terms ($l = 0$, 1, and 2).
Hint:
Explicit spherical harmonics are found in your lecture notes, in Jackson (p. 109), and in the *Particle Physics Booklet* (p. 266 in the 2002 edition). Note that

$$Y_{l\,-m} = (-1)^m Y^*_{lm}\ .$$

39.
Arrange five finite point charges at five different positions so that all $l \le 4$ moments of the charge distribution vanish, except for the $zzzz$ component of the hexadecapole moment

$$q_{40} \equiv \int d\tau' \rho(\vec{r}\,') r'^4 Y^*_{40}(\theta', \phi') \; .$$

Hint:
As noted in an earlier hint, the $m \neq 0$ moments vanish if the charge distribution is cylindrically symmetric about the z axis; for point charges this can be accomplished only by placing the charges on the axis. The $m = 0$, $l =$ odd moments vanish if the charge distribution is even in z. Place four charges away from the origin to cancel the $m = 0$ quadrupole ($l = 2$) moment; place a fifth charge at the origin to cancel the monopole ($l = 0$) moment.

40.
Consider the dimensionless operator

$$\vec{L} \equiv \tfrac{1}{i} \vec{r} \times \nabla$$

(apart from a missing factor of $\hbar$, this is the same as the angular momentum operator used in quantum mechanics).
(a.)
In spherical polar coordinates, show that

$$i\vec{L} = \hat{\phi}\frac{\partial}{\partial\theta} - \frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\phi} \; .$$

Hint:
Consult Griffiths' inside cover (GIC) #1.
(b.)
Express $\hat{\theta}$ and $\hat{\phi}$ in terms of $\hat{x}$, $\hat{y}$, $\hat{z}$, θ, and ϕ.
Hint:
Consult GIC #4.
(c.)
Show that

$$iL_z = \frac{\partial}{\partial\phi}$$

$$L_\pm \equiv L_x \pm iL_y = e^{\pm i\phi}\big(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\big) \; .$$

[$L_\pm$ are *raising* and *lowering* operators, which, within a factor, change Y_{lm} into $Y_{l,m\pm1}$.]

Hint:
Plug the result of (b.) into the result of (a.). To identify L_x, L_y, and L_z, collect the terms multiplying $\hat{x}$, $\hat{y}$, and $\hat{z}$.
(d.)
Show that

$$L^2 = L_z^2 + \tfrac{1}{2}\{L_+, L_-\} \; ,$$

where $\{a, b\}$ is the anticommutator $ab + ba$.
Hint:
Write down L_+L_- and L_-L_+ in terms of L_x and L_y (remember to preserve the order of the operators).
(e.)
Finally, show that

$$-L^2 = r^2\nabla^2_{\rm ang} \; ,$$

where $\nabla^2_{\rm ang}$ is the part of ∇^2 which involves derivatives in θ and ϕ.
Hint:
Plug the results of (c.) into the result of (d.). Consult GIC #1.

41.
Starting from the orthonormality of the spherical harmonics,

$$\int d\Omega \, Y^*_{l'm'}(\theta,\phi)\, Y_{lm}(\theta,\phi) = \delta_{ll'}\delta_{mm'} \; ,$$

and using the properties of $\vec{L}$ established in the previous problem, prove that the *vector* spherical harmonics

$$\vec{X}_{lm}(\theta,\phi) \equiv \vec{L}\, Y_{lm}(\theta,\phi)$$

satisfy the normality condition

$$\int d\Omega \, \vec{X}^*_{l'm'}(\theta,\phi)\cdot\vec{X}_{lm}(\theta,\phi) = l(l+1)\delta_{ll'}\delta_{mm'} \; .$$

Hint #1:
Write $\vec{L} = \hat{x}L_x + \hat{y}L_y + \hat{z}L_z$ and express L_x and L_y in terms of the raising and lowering operators L_+ and L_-. Use their standard properties

$$L_+ Y_{lm} = \sqrt{(l+m+1)(l-m)}\, Y_{l\ m+1}$$
$$L_- Y_{lm} = \sqrt{(l-m+1)(l+m)}\, Y_{l\ m-1}\ .$$

Collect terms in $Y^*_{l\ m-1} Y_{l\ m-1}$, $Y^*_{lm} Y_{lm}$, and $Y^*_{l\ m+1} Y_{l\ m+1}$, and exploit the orthonormality of each spherical harmonic. Note that $\delta_{m-1\ m'-1} = \delta_{mm'} = \delta_{m+1\ m'+1}$.

Hint #2:
For a more elegant solution, you may use the *Hermitian conjugate* $\vec{L}^\dagger$ of the angular momentum operator, defined by

$$\int d\Omega\, f^*(\Omega) \vec{L}\, g(\Omega) \equiv \int d\Omega \left(\vec{L}^\dagger f^*(\Omega) \right) g(\Omega)\ ,$$

where f and g are any two functions of Ω. Because $\vec{L}$ corresponds to a physical observable ($\hbar^{-1}$ times the angular momentum), it is *self-conjugate* ($\vec{L} = \vec{L}^\dagger$), or *Hermitian*.